Optimal Auction Design in a Multi-unit Environment: The Case of Sponsored Search Auctions

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ABSTRACT
We characterize the optimal (revenue maximizing) auction for sponsored search advertising. We show that a search engine's optimal reserve price is independent of the number of bidders. Using simulations, we consider the changes that result from a search engine's choice of reserve price and from changes in the number of participating advertisers.

Categories and Subject Descriptors
Economics

General Terms
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Keywords
Generalized Second Price auction, GSP, reserve price, search engine advertising

1. INTRODUCTION
The Generalized Second Price Auction (GSP) is a widely used mechanism for selling advertisements on Internet search engines. Each time a user enters a search term into a search engine, a GSP-type auction allocates the advertising space within that user's search results. There are hundreds of millions of separate GSP auctions conducted every day.

1.1 Sponsored Search Auctions Generally
Our analysis considers a simplified model of sponsored search auctions. We assume that each advertiser knows its value per click. We take all advertisers to have the same click-through rate (CTR) in a given position, and we assume that CTR's by position are common knowledge. Advertisers' marginal utility per click is non-decreasing in the number of clicks, and all advertisers maximize expected profit (defined as the total value of clicks received minus total payments in the auction).

Our analysis considers GSP auctions with reserve prices. Only advertisers who bid at least the reserve price are allowed to participate in an auction. Within a given keyword market, the lowest bidder pays the search engine's reserve price. In contrast, for each advertiser other than the lowest, the advertiser's per-click payment results from the bid of the advertisers immediately below: The n^{th} highest bidder pays the bid of $n+1^{st}$ bidder.[1] [9] gives further details on the GSP mechanism.

Search engines' revenues from GSP-type auctions are on the order of ten billion dollars per year. As a result, these advertising auctions are receiving considerable attention from practitioners and academics. For example, [1] considers the role of bid increment, [10] and [13] consider the implications of ranking rules, and [1], [5], and [14] consider the effect of budgets. [4] and [16] use simulations to study sponsored search auctions.

1.2 Our Contribution
This paper studies two important aspects of GSP auctions: the role of reserve price and the role of market depth. As far as we know, neither of the two has been investigated in the previous literature. First, in section 2, we study the role of optimal reserve prices. We show that a search engine's optimal reserve price is independent of the number of bidders. We then proceed to simulations to measure the benefits of optimal reserve prices in a variety of market conditions.

Second, in section 3, we consider the effect of "market depth" – the number of competing advertisers – on search engine revenues. All else equal, the more advertisers that bid for a particular keyword, the higher are search engine revenues for that keyword. Our simulations let us measure this effect. We then separate out the two effects by which arrival of a new advertiser increases a search engine's revenues: The new advertiser makes payments of his own (the direct effect), and the new advertiser spurs other advertisers to increase their bids (the indirect effect).

We consider both reserve prices and market depth in part because these seemingly-disparate topics share an underlying endogeneity: We show that, in both cases, a change has a first-order indirect effect. Furthermore, in the context of single-unit auctions, [7] shows that the

[1] In search engines' implementation, the n^{th} bidder plays the bid of the $n+1^{st}$ bidder plus a minimum bid increment of \$0.01. We set that increment to \$0 for expositional clarity.

incremental revenue from attracting an additional bidder is at least as large as the revenue increase from using the optimal reserve price. Our simulations indicate that with realistic parameter values, this result does not hold in GSP multi-unit auctions.

2. RESERVE PRICES AND MARKET OUTCOMES

Even after choosing other market parameters (e.g. use of GSP rather than a first-price mechanism), search engines can adjust their reserve price with relative ease. For instance, Yahoo! previously increased reserved price from 5 cents to 10 cents in most markets. Google now uses variable reserve price. [12] Nonetheless, we know of no theoretical analysis of optimal reserve prices in sponsored search markets. So profit-seeking search engines may reasonably wonder: What reserve price maximizes expected revenues?

In general, an optimal mechanism is a mechanism that maximizes the expected revenues of the seller. In some markets, optimal mechanisms are well-known. But the auction for search advertisements is a multi-unit auction, and optimal mechanism design in multi-unit auctions are an open problem.[2] Recently [17] characterized optimal auctions for a class of environments where bidders have one-dimensional types. We combine that result with equilibrium analysis of [9] to show that the GSP auction with an optimally-chosen reserve price is an optimal mechanism. Furthermore, in Proposition 1, we show how to find the optimal reserve price.

We then proceed to simulations. We decompose a search engine's benefit from optimal reserve prices into two components: First, the lowest-bidding advertiser's payment increases penny-for-penny with the reserve price. This is the direct effect of an increased reserve price. Second, the lowest bidder's increased bids spur other advertisers to increase their payments in turn, as detailed in section 2.4. This is the indirect effect of the reserve price. We proceed by developing a simulation methodology to predict these effects as a function of market primitives. We then compare the relative size of the direct and indirect effects.

2.1 A Numerical Example

Before we proceed to estimation of the benefit of optimal reserve prices on search engine revenues, we offer a numerical example to acquaint readers with the mechanics of GSP bidding as well as to provide greater intuition on the effects of reserve prices.

Consider an auction with two advertisers and two slots. Suppose the top slot yields 300 clicks per hour, and the bottom slot 200. Advertiser A values a click at $1, while B values a click at $0.70. The reserve price is $0.10.

Following [9], we compute that the envy-free bid of advertiser B is $0.30. To see why, consider B's perspective on possible changes of A's bid. If A were to revise his bid to fall ε below B's bid, B would pay his own bid ($0.30), and he would move into first position, where he would receive 300 clicks per hour. B would then realize hourly surplus of (300)($0.70-$0.30)=$120. But B gets exactly this same payoff in the second position with a payment of $0.10 (the reserve price), because (200)($0.70-$0.10)=$120 also. So B is indifferent between the two outcomes – exactly the envy-free concept from [9].

Now suppose the reserve price increases to $0.40. Then B's envy-free point increases to $0.40. $0.40 is the envy-free bid because (300)($0.70-$0.50)=200($0.70-$0.40)=$60.

Notice that the increase in reserve price has two distinct effects. First, since B remains in the lowest position (where payment equals the reserve price), B's payment increases from $0.10 per click to $0.40 per click. So B's total payment increases from $20 to $80. Second, A's per-click payment changes (itself set by B's bid) increases in the same way that B's bid increases, namely from $0.30 to $0.50. A's total payment therefore increases from $90 to $150.[3]

2.2 Simulation Methodology

To test the effects of search engines' reserve prices, we run simulations on a range of advertiser valuations, and we measure the search engine revenues that result from each reserve price regime. When considering optimal reserve prices, we let a search engine select the reserve price with highest expected revenues (taken across a range of simulation valuations from a given distribution). By comparing search engine revenues under no-reserve, low-reserve, and optimal reserve regimes, we can assess how reserve prices affect revenues under a variety of market conditions.

Our simulations require a model of advertiser equilibrium behavior. Such a model lets us predict advertiser bids (hence payments and ultimately search engine revenues) as a function of simulated valuations drawn from a distribution of valuations. For this purpose, we use the

[2] Myerson's [15] proves that adding a reserve price to otherwise efficient second price auction is an optimal mechanism in the case of symmetric bidders. But Myerson's result does not extend to multi-unit auctions in general. The optimal mechanism design in multi-unit auctions remains an open problem. See [8] for recent bounds on revenues in multi-unit auctions.

[3] In this example, we choose numerical values where increasing the reserve price yields identical increases in the bids of both bidders. In general, that is not the case.

equilibrium concept developed in [9]. The idea is that in equilibrium each advertiser bids at an envy free point—a point where an advertiser is exactly indifferent between remaining in his current position and trading places with a bidder above him. In this equilibrium, an increase in reserve price can lead to changes in bids of all advertisers, not just advertisers that were previously bidding below the new reserve price.

Except where otherwise indicated, our simulations use the following parameters: 5 advertisers; advertiser valuations drawn from a log-normal distribution with mean 1 and standard deviation 0.25; $0.10 reserve price; 1000 simulation iterations. Throughout, we assume that click through rate (CTR) in each position are common knowledge.[4]

2.3 Reserve Price and Search Engine Revenue

Search engines can set reserve prices to increase their revenues. Consider Figure 1, showing simulations of a market with five advertisers, with valuations drawn from a log-normal distribution.

The bell-shaped curve in Figure 1 represents the search engine revenues as a function of reserve price. With these market parameters, the search engine's optimal reserve price is $0.73. With optimal reserve price the search engine can increases its revenue by 68% relative to the case of zero reserve price and by 52% relative to $0.10 reserve price. Such an increase offers a major benefit to the search engine – albeit at a cost to advertisers, in that advertiser surplus falls 59% when the search engine sets its optimal reserve (rather than no reserve).

So long as a search engine keeps its reserve price below the valuation of the lowest bidder, increases in the search engine's reserve price only affect a transfer from advertisers to search engine, but do not affect total surplus. Notice that total surplus remains flat through the reserve price of approximately 0.6. However, if the reserve price exceeds the valuation of one or more bidders, bidders drop out, reducing total surplus.

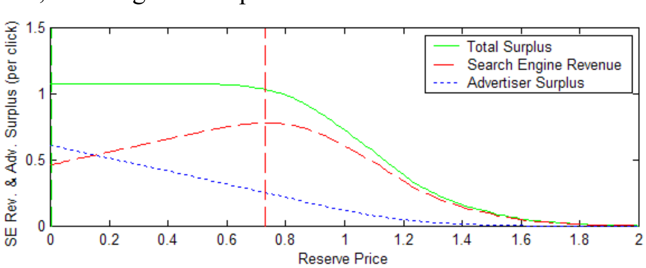

Figure 1. Per-click search engine revenue, advertiser surplus, and total surplus as a function of reserve price

[4] We use estimated CTRs from [6], to the extent available. Beyond the range reported in [6], we assume that CTRs decay geometrically at the same rate as the average in [6].

2.4 Advertiser Payments

When a search engine increases its reserve price, the first advertisers affected are those whose bids fall below the new minimum. But other advertisers are affected too.

Consider the response of the second-lowest advertiser as the reserve price increases. The lowest-ranked advertiser is increasing his payment as the reserve price rises, and eventually the lowest-ranked advertiser will observe that with only a small further increase in his bid, he could become second-lowest rather than lowest. To discourage such a jump by the lowest-ranked advertiser, the second-lowest advertiser must increase his bid somewhat. Then the third-lowest advertiser must increase his bid too, and the increases cascade upwards, with a nonzero impact on even the top advertiser.

Figure 2 shows these effects. The advertiser shown in the top-most plot is the advertiser with the highest valuation, who is allocated the highest position and makes the highest payment per click (no matter the search engine's reserve price). Lower-ranked advertisers fall below. Notice penny-for-penny increase for the bottom-most advertiser, as the reserve price increases. Other advertisers' payments increase less sharply.

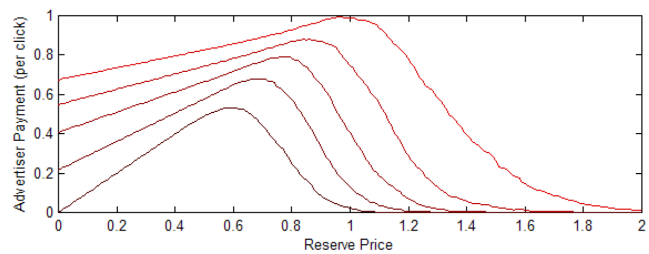

Figure 2. Advertisers' total payments as a function of the search engine's reserve price

As the reserve price increases, payments increase more sharply *on a percentage basis* for low-ranked advertiser than for higher advertisers. But higher-ranked advertisers receive far more clicks than lower-ranked advertisers, due to the greater prominence of top advertising positions. In general the *total increase in payment* from top-ranked advertisers is more than the total increase from low-ranked advertisers.

Figure 3 shows the total increase in each advertiser's payment, when a search engine sets its reserve price optimally versus when it sets a reserve price of $0.10. (Here again, simulations consider a market with five advertisers and log-normal valuations.) Comparisons with a reserve price of $0 are even more stark. Crucially, the lowest-ranked advertiser is *not* the hardest hit by the increase in reserve prices. Instead, higher-ranked advertisers end up facing a larger increase in payments, due to their larger volume of clicks.

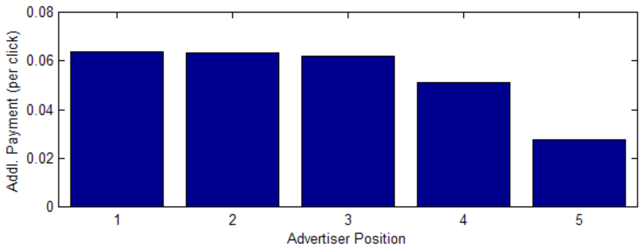

Figure 3. Total increase in each advertiser's payment, when reserve price is set optimally versus at $0.10

2.5 When Optimal Reserves Matter Most

Certain market conditions make search engines' choice of reserve price particularly significant, while other conditions render the choice of reserve price less important for search engine revenues.

Figure 4 shows the percent increase in search engine revenue (relative to revenue with a $0.10 reserve price) as a function of the number of advertisers participating in a given keyword market. A search engine's gain from setting an optimal reserve price is particularly large when few advertisers are bidding.

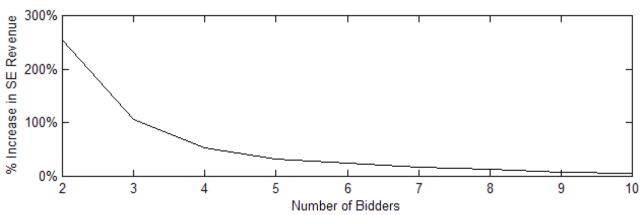

Figure 4. Percent increase in search engine revenue when search engines set optimal reserve prices

2.6 Optimal Mechanism

So far we described simulations that find optimal reserve prices for GSP auctions. In this section, we show that optimal reserve prices yield an efficient mechanism. We also show that the optimal reserve price depends only on the distribution from which bidder valuations are drawn – but not on the number of bidders or on the rate at which click-throughs decline from position to position.

Fully defining the GSP auction requires specifying the order of moves and the information structure. This section takes bidders' valuations to be private IID draws from a known distribution, and assumes that the formal structure of the game is as in [9] (section IV).

Let v_i denote the value of bidder i, and let α_j denote the CTR of position j. The value of position j to advertiser i is $\alpha_j v_i$. Assume that bidder values are IID draws from a distribution that satisfies the following regularity condition: $\frac{1-F(v)}{f(v)}$ is a decreasing function of v. Let v* denote the solution of $\frac{1-F(v)}{f(v)}=v$.

Proposition. *A GSP auction with a reserve price v* is an optimal mechanism.*

Proof. The proof is based on combining equilibrium analysis of GSP from [9] with the results obtained in [17]. ([17] is in turn a generalization of Myerson's classic [15].)

Following [17], we define the "generalized" virtual value of each bidder for each object (position). The generalized virtual value of advertiser i for position j is $w(j,i)=\alpha_j(v_i-\frac{1-F(v)}{f(v)})$. [17] shows that the optimal mechanism is an incentive-compatible mechanism that allocates objects so that the sum of the virtual values of all bidders is maximized.

It is easy to see that the optimal mechanism never allocates a position to an advertiser for whom virtual value, $v_i-\frac{1-F(v)}{f(v)}$, is negative. Generalized virtual value is proportional to virtual value, and α's are all positive. Hence allocating objects to agents with negative virtual values will reduce the sum of generalized virtual values. So the optimal mechanism only allocates positions to advertisers with nonnegative virtual values.

Now let us show that assortative matching between positions and agents with positive virtual values will maximize the sum of virtual values. It suffices to note that if an agent with higher value is ever placed below an agent with a lower value, exchanging positions of these agents would increase the sum of virtual values. Setting reserve price at v* guarantees that only bidders with positive virtual values will be allocated positions. (If the number of positions equals or exceeds the number of bidders with positive virtual valuations, all bidders with positive virtual values get positions allocated to them.)

It then remains to be shown that GSP with a reserve price can lead to equilibrium where bidders are assortatively sorted into positions based on values. It follows from the proof of Theorem 2 of [9] that GSP with reserve price implements an assortative matching between advertisers and positions.

Remark. A similar approach yields a *truthful* revenue-maximizing auction. The following modification of VCG is a truthful optimal mechanism. Payments are calculated as follows: With n slots available, introduce n+1 "shadow" bidders, each with a valuation of v*. Compute payments in the auction using the standard VCG formula, considering shadow bidders as if they were real bidders. This ensures that no bidder with a negative virtual value will be assigned a slot.[5]

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[5] The equivalence of this mechanism to GSP with reserve price follows from an argument identical to the one in [9].

3. MARKET DEPTH

We use the term "market depth" to refer to the number of advertisers vying for advertising associated with a given keyword. We say a keyword market is "deep" if that keyword is sought by many advertisers. Conversely, a market is "shallow" if it is of interest to only a few.

Search engines seem to recognize the importance of market depth in their continued recruitment of new advertisers. For example, Google offers a $20 reward whenever a participating web site refers a new AdWords advertisers. [11] Furthermore, both Google and Yahoo offer substantial signup bonuses to new advertisers. [1] Recruiting advertisers makes sense: Each advertiser brings a budget to be spent on search advertising, and search engines predictably seek access to these additional funds. But market characteristics make some new advertisers more valuable than others.

3.1 Simulation Setup

Our simulation methodology lets us investigate the effect of market depth on search engine revenues.

Consider the effect on search engine revenues when the number of bidders increases by one, i.e. from n to $n+1$. Of course, this effect depends on the valuation of the new bidder. We consider two distinct approaches. First, we let the new bidder draw his valuation from the same distribution as the existing bidders. Such a new bidder is identical to the others, in expectation, because all draws are IID. Second, we order a market's n advertisers from highest to lowest, and we add an $n+1^{st}$ advertiser with valuation of ε below the n^{th} advertiser. This approach reflects a view of the new bidder as a marginal bidder – one whose low valuation makes him least likely to participate.

3.2 Market Depth and Search Engine Revenue

The incremental value of adding an advertiser declines as the number of bidders grows. Figure 5 shows this effect, plotting the marginal value of an additional advertiser (vertical axis) against the number of advertisers already present (horizontal axis). Separate plotted curves reflect advertisers with valuations drawn from the two distributions set out in section 3.1. The figure confirms that a marginal bidder provides greatest benefit to a search engine when the bidder participates in a market with few existing bidders.

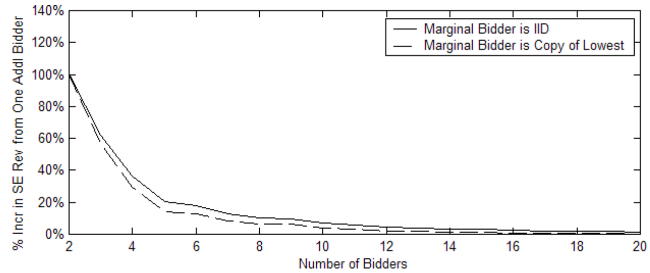

Figure 5. Percent increase in search engine revenue when a marginal bidder arrives, a function of the number of bidders initially present

3.3 Direct and Indirect Revenue Effects of Increasing Market Depth

When a new advertiser arrives, a search engine's revenue increases for two distinct reasons. First, the new advertiser makes payments for the advertising it purchases. This is the direct effect of the new advertiser's arrival. Second, the new advertiser causes an increase in competition for affected search terms, prompting other advertisers to increase their bids. This is the indirect effect of the new advertiser's arrival.

Our simulations let us compare the size of these effects. The result of the comparison depends on the distribution of valuations of the newly-arriving advertisers.

Figure 6 analyzes outcomes[6] when new advertisers are IID, drawn from the same distribution as other advertisers. The payments from the newly-arrived advertiser (the first bar in each group) remains relatively constant as the number of advertisers increases. But with more advertisers, the search engine receives less additional revenue when a new advertiser arrives. (Notice the downward trend in the third bar in each group). Furthermore, when the market is sufficiently deep, the arrival of a new advertiser actually *reduces* the sum of other advertisers' payments. (Notice the negative values of the second bar in each group.) These reductions reflect that the arrival of an additional advertiser leaves less advertising for others to buy. In short, when advertiser valuations are IID and when several advertisers are already present, the direct effect of a new advertiser's arrival is the dominant effect, while indirect effects are small.

[6] This section reports revenues "per search" rather than "per click" in order to account for the underlying endogeneity of clicks. As more advertisers arrive, more ads can be shown (i.e. in space otherwise left blank), and hence more clicks can occur. We seek to minimize this effect, in order to focus on revenues resulting from competition among advertisers We therefore make the conservative assumption that the CTR of the first position is independent of the number of ads shown on the page.

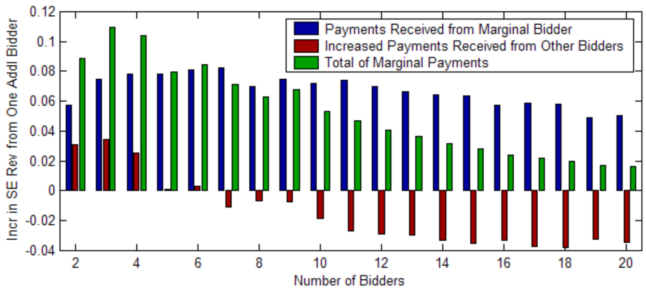

Figure 6. Direct and indirect per-search revenue effects of a new advertiser – when new advertisers are IID

In contrast, Figure 7 analyzes the case in which new advertisers have systematically low valuations. Here, the arrival of a new advertiser tends to have little direct effect, because the advertiser always takes the lowest position and pays the minimum bid. But the new advertiser's presence nonetheless causes other advertisers to increase their bids – an indirect effect that our simulations indicate can be relatively large.

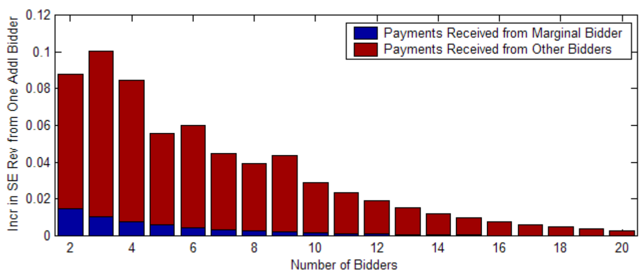

Figure 7. Direct and indirect per-search revenue effects of a new advertiser – when new advertisers have low valuations

3.4 Market Depth versus Optimal Reserve

Comparing Figure 4 with Figure 5 lets us investigate the celebrated [7] result that adding another bidder yields more revenue than setting an optimal reserve price.

Indeed, our simulations suggest that adding another bidder is not always preferable to setting an optimal reserve, in this multi-unit context. For example, Figure 4 reports that, with five bidders, setting an optimal reserve price increases search engine revenues by approximately 30%. In contrast, Figure 5 shows that, with five bidders, an additional IID bidder increases search engine revenues by approximately 20% – less than 30%, contrary to the result in [7].

Importantly, the result from [7] need not hold in this context. [7] derives from properties of single-unit auctions, but here multiple advertising positions are available. With multiple positions available, adding another bidder need not bring as much competition as if only a single position were offered.

4. CONCLUSION

We have shown that a GSP auction with a reserve price is an optimal mechanism, and we have shown how to calculate the optimal reserve price. We have used simulations to evaluate when an optimal reserve price is most valuable, and we have compared the revenue benefits of an optimal reserve price to the benefits from a marginal bidder. We have separated these revenue effects into their direct and indirect components; we have shown that indirect components are first-order and often significantly larger than direct components.

Our simulations let us examine the effects of changes in market conditions. This is a flexible methodology, well-suited to a variety of questions and policy experiments.

5. ACKNOWLEDGMENTS

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